PERGAMON

International Journal of Solids and Structures 39 (2002) 2359–2360

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Discussion

Discussion on "A. Aguiar and R. Fosdick, Self-intersection in elasticity" [International Journal of Solids and Structures 38 (28) (2001) 4797–4825]

M. Rahman

Daimler-Chrysler Corporation, 800 Chrysler Drive, CIMS 484-14-01, Auburn Hills, MI 48326, USA

Received 19 September 2001; received in revised form 19 September 2001

This article as well as a recent article by Fosdick and Royer-Carfagni (2001) discuss an interesting phenomenon in linear elasticity. Specifically, it is observed that there are many problems in linear elasticity theory whose solutions guarantee the injectivity of the deformation field *not* everywhere in the domain. In such cases, the Jacobian of elastic deformation transformation becomes negative, i.e.

$$I(x) = \det\left[\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right] < 0, \tag{1}$$

where δ_{ij} is Kronecker's delta, $u_i(x)$ is the elastic displacement vector, and x denotes the triplet of Cartesian coordinates (x_1, x_2, x_3). Such a situation is, of course, related to inter-penetration of material particles.

These writers then illustrate the occurrence of this phenomenon in Abramov's bonded punch problem. However, these results are not new. They are obviously unaware of the fact that Savin and Rvachev had first discovered this phenomenon in 1963 (Savin and Rvachev, 1963a, 1963b, 1964). Their results are also summarized as a separate chapter in the book by Rvachev and Protsenko (1977). In particular, Savin and Rvachev showed that the Jacobian in Abramov's problem is given by

$$I(x_1, 0) = 1 - \frac{Pv}{\pi\mu\sqrt{\chi}\sqrt{l^2 - x_1^2}} \cos\left(\beta \log\frac{l + x_1}{l - x_1}\right), \tag{2}$$

where $\beta = \log\chi/2\pi$, $\chi = 3 - 4v$ (v is the Poisson ratio of the material of the half-space). As can be seen from Eq. (2), the inequality $I(x_1, 0) > 0$ is violated infinite number of times as the punch corner is approached. Away from the punch corner, there is no such anomalous behavior.

My own interest (1990) in their results was aroused by the problem of a punch moving across an elastic half-space. It is well known that there are problems when the punch moves at a transonic speed, which cannot be resolved within the scopes of classical linear elasticity theory. Since the governing integral equation for this problem resembles that corresponding to Abramov's bonded punch problem, I anticipated that the anomalous behavior of the contact stresses under a punch moving at a transonic speed might be related to Savin and Rvachev's observation. Several years later, in view of the fact that their observation

PII: S0020-7683(01)00281-5

remained unnoticed to the western elasticity community, I wrote an article (1999) slightly expanding their results.

Essentially, what Aguiar and Fosdick call the lack of local injectivity, Savin and Rvachev interpret it as the violation of *factual compatibility* of deformations. Apart from Abramov's problem, these authors illustrated the occurrence of this phenomenon in several other problems. One of these problems is that of Boussinesq. As they showed (see also Rahman (1999)), even in this problem the factual compatibility of deformation is guaranteed not everywhere in the domain. In fact, in this problem, the zone of fictitious compatibility (that is, the zone where the injectivity of deformation is violated) is a body of revolution formed by a plane figure around x_3-axis. It is therefore not without foundation to assume that the loss of factual compatibility might be also present in many mixed boundary value problems of linear elasticity, since the point force solution enters into the governing integral equations for these problems as kernels. Indeed, as we see, this is the case with Abramov's bonded punch problem. Some other problems sharing the same type of symptoms include the axisymmetric sharp punch problem and the problem of a tensile mode circular crack (see Rahman (1999)). Remarkably, Savin and Rvachev also proposed a remedy to this problem, which consists of using a *modified* point force solution that does not have this spurious behavior (for details, see Rahman (1999)). However, the price for this modification is very high in that the modified point force solution becomes *non-linear* and as such we are deprived of using the superposition principle to formulate the mixed boundary value problems in the form of integral equations. Such non-linear problems can be solved in an incremental fashion only. Interestingly, the results of Aguiar and Fosdick seem to corroborate this point as they find that their modified semi-linear material does not share this anomalous behavior and that it satisfies Eq. (1) everywhere in the vicinity of the punch corner. Essentially, the same conclusion was also reached in Rahman (1999) based on Savin and Rvachev's observation and Knowles and Sternberg's results (1975).

Despite the overlaps, the authors should be commended for the interesting numerical results presented in the article. Furthermore, in Fosdick and Royer-Carfagni (2001), laudable attempts have been made to rectify this drawback, which certainly constitutes a gap in linear elasticity by reformulating such problems as the minimization ones with the constraint (1). The proof of the existence and uniqueness of the minimizers that these authors presented is truly commendable.

References

Fosdick, R., Royer-Carfagni, G., 2001. The constraint of local injectivity in linear elasticity theory. Proc. Roy. Soc. Lond. A 457, 2367–2387.

Savin, G.N., Rvachev, V.L., 1963a. On the violation of compatibility of deformation in some problems of the theory of elasticity. Dokl. Ukrainian SSR 3, 354–357 (in Russian).

Savin, G.N., Rvachev, V.L., 1963b. On formal and factual compatibility of deformations. Dokl. Ukrainian SSR 1, 35–39 (in Russian).

Savin, G.N., Rvachev, V.L., 1964. On the displacement under a concentrated force. Prikladnaya Mekhanika 10 (2), 222–225 (in Russian).

Rvachev, V.L., Protsenko, V.S., 1977. Contact Problems of the Theory of Elasticity for Non-Classical Domains. Naukova Dumka, Kiev (in Russian).

Rahman, M., 1999. The Savin–Rvachev observation on violation of factual compatibility in linear elastostatics and some of its implications. Int. J. Eng. Sci. 37, 1277–1288.

Knowles, J.K., Sternberg, E., 1975. On the singularity induced by certain mixed boundary conditions in linearized and nonlinear elastostatics. Int. J. Solids Struct. 11, 1173–1201.